

Formula for Vector Rotation in Arbitrary Planes in \mathfrak{R}^n

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1 Introduction

We derive a formula for the rotation of a vector in an arbitrary plane that is applicable to \mathfrak{R}^n for all $n \geq 2$. Since all general rotations can be decomposed into a sequence of plane rotations, this formula is applicable to rotations in general.

Although in \mathfrak{R}^3 every plane of rotation is uniquely defined by an axis, this is not true in any other dimension. Hence, a better way of defining rotation is by specifying the *plane* in which we are rotating, and a point on this plane, the *center of rotation*, about which we are rotating.

Formulae for vector rotation are well-known. Here, however, we desire a formula applicable to \mathfrak{R}^n for all $n \geq 2$, and therefore need a formulation purely in terms of vector operations, and not involving any explicit Cartesian coordinates.

2 Assumptions

Let $\mathbf{v} \in \mathfrak{R}^n$ be a vector. We shall denote the image of \mathbf{v} under rotation in a plane P by an angle of θ as $\text{rot}_{P,\theta}(\mathbf{v})$. Without loss of generality, we make the following assumptions:

1. The actual plane of rotation P is parallel to a plane P_0 that passes through the origin.

2. The center of rotation C lies in the $(n-2)$ -hyperplane that intersects P_0 at the origin. That is, the desired rotation in P about C is isomorphic via translation to a rotation in the plane P_0 about the origin.
3. P_0 is the span of two orthogonal unit vectors $\mathbf{x}, \mathbf{y} \in \mathfrak{R}^n$. Since planes of rotation are always 2-dimensional, \mathbf{x} and \mathbf{y} are sufficient to fully specify P_0 .
4. We define the sense of rotation as rotating \mathbf{x} into \mathbf{y} . That is,

$$\text{rot}_{P_0, \pi/2}(\mathbf{x}) = \mathbf{y}$$

3 Derivation

3.1 Reduction to Rotation in P_0

Let θ be the angle by which we wish to rotate \mathbf{v} . We shall achieve the rotation by rotating the projection of \mathbf{v} in P_0 , and then mapping the result back to plane in which \mathbf{v} lies.

Let \mathbf{v}_p be the projection of \mathbf{v} onto P_0 :

$$\mathbf{v}_p = (\mathbf{v} \cdot \mathbf{x})\mathbf{x} + (\mathbf{v} \cdot \mathbf{y})\mathbf{y} \tag{1}$$

Theorem 1 $(\mathbf{v} - \mathbf{v}_p)$ is orthogonal to P_0 .

Proof. We show that $(\mathbf{v} - \mathbf{v}_p)$ is orthogonal to \mathbf{x} and \mathbf{y} .

$$\begin{aligned} (\mathbf{v} - \mathbf{v}_p) \cdot \mathbf{x} &= \mathbf{v} \cdot \mathbf{x} - \mathbf{v}_p \cdot \mathbf{x} \\ &= \mathbf{v} \cdot \mathbf{x} - ((\mathbf{v} \cdot \mathbf{x})\mathbf{x} + (\mathbf{v} \cdot \mathbf{y})\mathbf{y}) \cdot \mathbf{x} \\ &= \mathbf{v} \cdot \mathbf{x} - ((\mathbf{v} \cdot \mathbf{x})(\mathbf{x} \cdot \mathbf{x}) + (\mathbf{v} \cdot \mathbf{y})(\mathbf{y} \cdot \mathbf{x})) \end{aligned}$$

But since \mathbf{x} and \mathbf{y} are orthogonal unit vectors,

$$\begin{aligned} (\mathbf{v} - \mathbf{v}_p) \cdot \mathbf{x} &= \mathbf{v} \cdot \mathbf{x} - ((\mathbf{v} \cdot \mathbf{x})\|\mathbf{x}\|^2 + 0) \\ &= \mathbf{v} \cdot \mathbf{x} - \mathbf{v} \cdot \mathbf{x} \\ &= 0 \end{aligned}$$

Similarly,

$$\begin{aligned}
(\mathbf{v} - \mathbf{v}_p) \cdot \mathbf{y} &= \mathbf{v} \cdot \mathbf{y} - \mathbf{v}_p \cdot \mathbf{y} \\
&= \mathbf{v} \cdot \mathbf{y} - ((\mathbf{v} \cdot \mathbf{x})\mathbf{x} + (\mathbf{v} \cdot \mathbf{y})\mathbf{y}) \cdot \mathbf{y} \\
&= \mathbf{v} \cdot \mathbf{y} - ((\mathbf{v} \cdot \mathbf{x})(\mathbf{x} \cdot \mathbf{y}) + (\mathbf{v} \cdot \mathbf{y})(\mathbf{y} \cdot \mathbf{y})) \\
&= \mathbf{v} \cdot \mathbf{y} - (0 + (\mathbf{v} \cdot \mathbf{y})\|\mathbf{y}\|^2) \\
&= \mathbf{v} \cdot \mathbf{y} - \mathbf{v} \cdot \mathbf{y} \\
&= 0 \quad \blacksquare
\end{aligned}$$

This means that $(\mathbf{v} - \mathbf{v}_p)$ is the component of \mathbf{v} that is orthogonal to P_0 . Hence, it is unchanged by the rotation in P . Therefore, the image of \mathbf{v} under rotation by θ in P is equal to the image of \mathbf{v}_p under rotation by θ in P_0 plus $(\mathbf{v} - \mathbf{v}_p)$. That is,

$$\text{rot}_{P,\theta}(\mathbf{v}) = \text{rot}_{P_0,\theta}(\mathbf{v}_p) + (\mathbf{v} - \mathbf{v}_p) \quad (2)$$

In other words, it is sufficient to compute the rotation of \mathbf{v}_p in P_0 by θ , and adding $(\mathbf{v} - \mathbf{v}_p)$ to the result.

3.2 Rotation in P_0

From (1), we see that the coordinates of \mathbf{v}_p in P_0 with respect to \mathbf{x} and \mathbf{y} are $\mathbf{v} \cdot \mathbf{x}$ and $\mathbf{v} \cdot \mathbf{y}$, respectively. Let ϕ be the angle between \mathbf{v}_p and \mathbf{x} . Then:

$$\mathbf{v} \cdot \mathbf{x} = \|\mathbf{v}_p\| \cos \phi \quad (3)$$

$$\mathbf{v} \cdot \mathbf{y} = \|\mathbf{v}_p\| \sin \phi \quad (4)$$

The resulting vector after rotating \mathbf{v}_p by θ makes an angle of $(\phi + \theta)$ with \mathbf{x} . Therefore:

$$\text{rot}_{P_0,\theta}(\mathbf{v}_p) = \|\mathbf{v}_p\| \cos(\phi + \theta)\mathbf{x} + \|\mathbf{v}_p\| \sin(\phi + \theta)\mathbf{y}$$

Using the trigonometric additive identities and applying (3) and (4), we have:

$$\begin{aligned}
\|\mathbf{v}_p\| \cos(\phi + \theta) &= \|\mathbf{v}_p\|(\cos \phi \cos \theta - \sin \phi \sin \theta) \\
&= (\|\mathbf{v}_p\| \cos \phi) \cos \theta - (\|\mathbf{v}_p\| \sin \phi) \sin \theta \\
&= (\mathbf{v} \cdot \mathbf{x}) \cos \theta - (\mathbf{v} \cdot \mathbf{y}) \sin \theta
\end{aligned} \quad (5)$$

And:

$$\begin{aligned}
\|\mathbf{v}_p\| \sin(\phi + \theta) &= \|\mathbf{v}_p\| (\sin \phi \cos \theta + \cos \phi \sin \theta) \\
&= (\|\mathbf{v}_p\| \sin \phi) \cos \theta + (\|\mathbf{v}_p\| \cos \phi) \sin \theta \\
&= (\mathbf{v} \cdot \mathbf{y}) \cos \theta + (\mathbf{v} \cdot \mathbf{x}) \sin \theta
\end{aligned} \tag{6}$$

Therefore, the image of \mathbf{v}_p under rotation in P_0 by θ is:

$$\begin{aligned}
\text{rot}_{P_0, \theta}(\mathbf{v}_p) &= ((\mathbf{v} \cdot \mathbf{x}) \cos \theta - (\mathbf{v} \cdot \mathbf{y}) \sin \theta) \mathbf{x} + \\
&\quad ((\mathbf{v} \cdot \mathbf{y}) \cos \theta + (\mathbf{v} \cdot \mathbf{x}) \sin \theta) \mathbf{y}
\end{aligned} \tag{7}$$

3.3 Rotation in P

Finally, substituting (1) and (7) into (2), we get:

$$\begin{aligned}
\text{rot}_{P, \theta}(\mathbf{v}) &= ((\mathbf{v} \cdot \mathbf{x}) \cos \theta - (\mathbf{v} \cdot \mathbf{y}) \sin \theta) \mathbf{x} + \\
&\quad ((\mathbf{v} \cdot \mathbf{y}) \cos \theta + (\mathbf{v} \cdot \mathbf{x}) \sin \theta) \mathbf{y} + \\
&\quad \mathbf{v} - ((\mathbf{v} \cdot \mathbf{x}) \mathbf{x} + (\mathbf{v} \cdot \mathbf{y}) \mathbf{y}) \\
&= \mathbf{v} + [(\mathbf{v} \cdot \mathbf{x}) \cos \theta - (\mathbf{v} \cdot \mathbf{y}) \sin \theta - (\mathbf{v} \cdot \mathbf{x})] \mathbf{x} + \\
&\quad [(\mathbf{v} \cdot \mathbf{y}) \cos \theta + (\mathbf{v} \cdot \mathbf{x}) \sin \theta - (\mathbf{v} \cdot \mathbf{y})] \mathbf{y} \\
&= \mathbf{v} + [(\mathbf{v} \cdot \mathbf{x})(\cos \theta - 1) - (\mathbf{v} \cdot \mathbf{y}) \sin \theta] \mathbf{x} + \\
&\quad [(\mathbf{v} \cdot \mathbf{y})(\cos \theta - 1) + (\mathbf{v} \cdot \mathbf{x}) \sin \theta] \mathbf{y}
\end{aligned} \tag{8}$$

Equation (8) gives us a formula for rotating the vector \mathbf{v} by θ in the plane defined by \mathbf{x} and \mathbf{y} . Since it is written entirely in terms of vector operations involving arbitrary-dimensional vectors \mathbf{v} , \mathbf{x} , and \mathbf{y} , it can be applied to all \mathfrak{R}^n for $n \geq 2$.

It can also be written in the following (notationally abusive) matrix form, which is visually more appealing:

$$\text{rot}_{P, \theta}(\mathbf{v}) = \mathbf{v} + \begin{bmatrix} \mathbf{x} & \mathbf{y} \end{bmatrix} \begin{bmatrix} (\cos \theta - 1) & -\sin \theta \\ \sin \theta & (\cos \theta - 1) \end{bmatrix} \begin{bmatrix} \mathbf{v} \cdot \mathbf{x} \\ \mathbf{v} \cdot \mathbf{y} \end{bmatrix} \tag{9}$$

In this form, its analogy with the familiar 2-dimensional rotation matrix is clear. The extra -1 terms are because \mathbf{v} appears as a separate term in the equation. When \mathbf{v} lies on the plane P_0 , the equation reduces to the familiar 2-dimensional form.